

UNIFORMLY DISTRIBUTED ELEMENTS ON METABELIAN LIE RINGS

E. N. POROSHENKO, E. I. TIMOSHENKO

ABSTRACT. In this paper, the notion of a uniformly distributed element on the variety of metabelian Lie algebras is introduced. This notion is analogous to one of a measure preserving element on group varieties. As the main result of the paper it was shown that on the variety of metabelian Lie algebras an element is primitive iff it is uniformly distributed.

1. INTRODUCTION

In this paper, we describe elements that uniformly distributed on the variety of Lie rings (over the ring of integers \mathbb{Z}). In [5, 6], elements uniformly distributed on groups were studied. Such elements are called measure preserving. In the papers [13, 14] of the second author, the notion of a measure preserving element on a group variety was introduced and the measure preserving systems of elements on the varieties of nilpotent and metabelian groups were described. It turns out that such systems of elements are primitive i.e. they can be completed to the bases of a free group of the corresponding variety.

Not only the uniform distribution on groups and rings but also other distributions can be considered. Some interesting problems arise in this case. An example of such problem is to describe the sets of elements having the same distribution on a given variety. For this reason, we think that it is appropriate to use the term “distribution of elements” and although the terms “measure preserving elements” and “uniformly distributed elements” have the same meaning we will use the second one.

The main result of the paper is Theorem 3.8. It claims that an element of a free metabelian Lie ring is uniformly distributed on the variety of metabelian Lie rings iff it is primitive. To prove this theorem we had to find a primitivity criterion for this ring variety (Theorem 3.4). The proof of this criterion is similar to one for the varieties of metabelian groups and metabelian Lie algebras, (see. [7, 8, 11, 12, 15, 16]) but it has some distinctions.

2. PRELIMINARIES

In this paper, we assume that $X = \{x_1, x_2, \dots, x_n\}$ is a finite set. By L denote the free Lie ring with the set of free generators X and by M denote the free metabelian Lie ring with the same set of generators, i.e. free and free metabelian Lie \mathbb{Z} -algebras respectively.

Let S be a Lie ring with generators x_1, x_2, \dots, x_n . By $U(S)$ denote the universal enveloping \mathbb{Z} -algebra of S .

We denote Lie monomials by bracketed lowercase Latin characters and associative monomials by lowercase Latin characters without brackets. Moreover, we use brackets to denote the images of Lie monomials by the natural map from S to $U(S)$.

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Namely, $[v]$ in $U(S)$ is a homogeneous associative polynomial obtained by complete removal of brackets by the rule $[[u_1], [u_2]] = [u_1][u_2] - [u_2][u_1]$.

Let $[v]$ be a Lie monomial. By $\ell([v])$ denote the length of this monomial i.e. the total number of entries of all generators in $[v]$.

Any element g in M can be considered as a Lie polynomial in the variables x_1, x_2, \dots, x_n . So, we denote it by $g(x_1, x_2, \dots, x_n)$. Although a representation of an element of M is not unique, we will see below that one can use any such representation.

Definition 2.1. An element g is said to be a *primitive* element of the metabelian Lie ring M , if it can be included in a system of free generators of this ring.

Let S^t be the direct sum of t copies of S , where S is an arbitrary Lie ring. Namely $S^t = \underbrace{S \oplus S \oplus \dots \oplus S}_{t \text{ times}}$. By p^t denote an t -tuple (p_1, p_2, \dots, p_t) of elements in S (then $p^t \in S^t$).

Let R be a finite Lie ring of some variety \mathfrak{M} . For an arbitrary g in the free Lie ring with n generators define the map $\psi_g : R^n \rightarrow R$ such that $\psi_g(r^n) = g(r_1, r_2, \dots, r_n)$. Namely, to obtain ψ_g one should substitute r_1, r_2, \dots, r_n for x_1, x_2, \dots, x_n in any representation of g as a Lie polynomial. Since any map of generators of a free Lie ring $L_{\mathfrak{M}}$ in a variety \mathfrak{M} to an arbitrary Lie ring of this variety can be extended to a homomorphism uniquely the value of $g(r^n)$ depends on the element $g(x_1, x_2, \dots, x_n)$ in $L_{\mathfrak{M}}$ but not on a polynomial representing this element. Consider the uniform distribution on R^n . Namely, suppose that each element r_1, r_2, \dots, r_n is chosen independently with probability $|R|^{-1}$. Then for any $r^n \in R^n$ probability to choose it is equal to $|R|^{-n}$.

Definition 2.2. Let \mathfrak{M} be an arbitrary variety of Lie rings and let R be a finite Lie ring in this variety. The element g of a free Lie ring in a variety \mathfrak{M} is called *uniformly distributed on R* , if for any $p \in R$ probability that $\psi_g(r^t) = p$ is equal to $|R|^{-1}$, where r^t is chosen at random. It means that if r^n runs over R^n then any $p \in R$ is the image of exactly $|R|^{n-1}$ elements of R^n .

Definition 2.3. An element g of a free Lie ring of a variety \mathfrak{M} is *uniformly distributed on \mathfrak{M}* if it is uniformly distributed on any finite Lie ring of this variety.

Clearly, the property of an element to be uniformly distributed on the variety of metabelian Lie algebras does not depend on a set of free generators chosen in M . Indeed, let y_1, y_2, \dots, y_n be some other system of free generators of M . Then we have

$$(1) \quad \begin{array}{ll} x_1 = f_1(y_1, y_2, \dots, y_n); & y_1 = h_1(x_1, x_2, \dots, x_n); \\ x_2 = f_2(y_1, y_2, \dots, y_n); & y_2 = h_2(x_1, x_2, \dots, x_n); \\ \dots\dots\dots & \dots\dots\dots \\ x_n = f_n(y_1, y_2, \dots, y_n). & y_n = h_n(x_1, x_2, \dots, x_n). \end{array}$$

Here f_i and h_j are Lie polynomials. Let

$$\hat{g}(y_1, \dots, y_n) = g(f_1(y_1, \dots, y_n), \dots, f_n(y_1, \dots, y_n)),$$

and

$$\check{g}(x_1, \dots, x_n) = \hat{g}(h_1(x_1, \dots, x_n), \dots, h_n(x_1, \dots, x_n)).$$

Then, obviously, $g(x_1, \dots, x_n) = \hat{g}(y_1, \dots, y_n) = \check{g}(x_1, \dots, x_n)$ in M .

Let $\mu : R^n \rightarrow R^n$ take each $r^n \in R^n$ to $(s_1, \dots, s_n) = s^n \in R^n$, where $s_i = f_i(r^n)$. Let us show that different r^n corresponds to different s^n . If it is not the case then the images of some two elements under μ coincide. Since the images of the elements in R^n are in R^n there exists an n -tuple $\check{s}^n = (\check{s}_1, \dots, \check{s}_n)$ not lying in the image of μ . Let $\check{r}_i = h_i(\check{s}^n)$. Then (1) implies $\check{s}_i = f_i(\check{r}^n)$, we get a contradiction.

Consequently, μ is a bijection thus for any $p \in R$ the number of n -tuples $r^n \in R^n$ such that $g(r^n) = p$ in R is equal to the number of n -tuples $s^n \in R^n$, such that $\hat{g}(s^n) = p$ in R .

The group analogues of uniformly distributed elements are called measure preserving elements (see [13]). Later on we will need the lemma from [13]. Since we need this lemma in a particular case let us provide its formulation in that case only.

Lemma 2.4. *Let A_n be a free abelian group of rank n . An element v of this group preserves measure on the variety of abelian groups iff it is primitive.*

Definition 2.5. For any associative commutative ring S a vector $(s_1, s_2, \dots, s_k) \in S^k$ is called *unimodular* if the ideal generated by the coordinates of this vector coincides with S .

The following definition generalizes Definition 2.5.

Definition 2.6. Let S be any associative commutative ring and let I be an ideal in this ring. A vector $(s_1, s_2, \dots, s_k) \in S^k$ is called *I -modular* if the coordinates of this vector generate I (as an ideal).

Let us formulate one more statement we will need in this paper.

Theorem 2.7. [2] *Let $S_0 \subset S_1 \subset \dots \subset S_r \subset \dots$ be a chain of commutative rings satisfying the following properties.*

- (1) *For any r the unit of S_r lies in S_0 .*
- (2) *Any ring S_r is a retract of S_{r+1} and the kernel of this retract is generated by an element y_{r+1} is not a zero divisor of the ring $\bigcup_i S_i$.*
- (3) *For any $t \geq r$ the group $GL(t, S_r)$ of invertible matrices of order t acts transitively on the set of unimodular vectors in S_r^t .*
- (4) *If I_r be the ideal in S_r generated by y_1, y_2, \dots, y_r , then I_r/I_r^2 is a free S_0 -module of rank r .*

Then for all $t \geq r$ the group $GL(t, S_r)$ acts transitively on the set of I_r -modular vectors in S_r^t .

Let us remind the definition of partial derivatives in free and free metabelian Lie rings. Consider the image of $f \in L$ in $U(L)$ under the natural embedding. For the sake of simplicity let us denote this image also by f . It is clear that there are unique elements $\partial'_1 f, \partial'_2 f, \dots, \partial'_n f \in U(L)$ such that

$$(2) \quad f = x_1 \partial'_1 f + x_2 \partial'_2 f + \dots + x_n \partial'_n f.$$

These elements are called *partial (right) derivatives* of f . Evidently, we can say that the maps $\partial'_i : L \rightarrow U(L)$ are derivations. Indeed, these maps have the following properties

$$\partial'_i(f + g) = \partial'_i f + \partial'_i g; \quad \partial'_i[f, g] = (\partial'_i f)g - (\partial'_i g)f$$

(more precisely the second property should be written as

$$\partial'_i[f, g] = (\partial'_i f)\omega(g) - (\partial'_i g)\omega(f),$$

where $\omega : L \rightarrow U(L)$ is the natural embedding of the free Lie ring L to $U(L)$, i.e. $\omega(x_i) = x_i$ and by induction $\omega([u], [v]) = \omega([u])\omega([v]) - \omega([v])\omega([u])$).

Let us define partial derivatives on a free metabelian Lie ring. By $\mathbb{Z}[X]$ denote the set of commutative associative polynomials in the variables x_1, x_2, \dots, x_n . Let $\varphi : L \rightarrow M$ be the natural homomorphism i.e. the homomorphism taking each x_i to itself and let $\varphi' : U(L) \rightarrow \mathbb{Z}[X]$ also be the natural homomorphism (that is $\varphi'(x_i) = x_i$). Consider the maps $\partial_i = \varphi' \circ \partial'_i \circ \varphi^{-1} : M \rightarrow \mathbb{Z}[X]$. It is easy to see that ∂_i are well defined. Indeed, the only difficulty in defining this map is that φ is not an isomorphism if $|X| > 1$. Therefore, each element $g \in M$ has more than one

pre-image under φ L . However, if $g_1, g_2 \in L$ are such that $\varphi(g_1) = \varphi(g_2) = g$ then $g_2 - g_1 \in \text{Ker } \varphi = \langle [L', L'] \rangle$, where L' is the derivative ring of L . Therefore, we have $g_2 - g_1 = \sum_j \alpha_j [[u_j], [v_j]]$, where $[u_j]$ are monomials in $[L', L']$, $[v_j]$ are monomials in L , and $\alpha_j \in \mathbb{Z}$. So, we obtain

$$\begin{aligned}
 \varphi' \circ \partial'_i(g_2 - g_1) &= \varphi' \circ \partial'_i \left(\sum_j \alpha_j [[u_j], [v_j]] \right) \\
 &= \varphi' \left(\sum_j \alpha_j \partial'_i([u_j], [v_j]) \right) \\
 (3) \quad &= \varphi' \left(\sum_j \alpha_j (\partial'_i([u_j])[v_j] - \partial'_i([v_j])[u_j]) \right) \\
 &= \sum_j \alpha_j (\varphi' \circ \partial'_i([u_j])\varphi'([v_j]) - \varphi' \circ \partial'_i([v_j])\varphi'([u_j])).
 \end{aligned}$$

Next, let $[w]$ lie in L' . Then $\varphi'([w]) = 0$. Indeed, $[w] = [[w_1], [w_2]]$ for some Lie monomials $[w_1], [w_2]$ in L consequently $\varphi'([w]) = \varphi'([w_1])\varphi'([w_2]) - \varphi'([w_2])\varphi'([w_1])$. If $[w] = [[w_1], [w_2]]$ lies in $[L', L']$ then $\partial'_i([w]) = \partial'_i([w_1])[w_2] - \partial'_i([w_2])[w_1]$. Since $[w_1], [w_2] \in L'$ we obtain as above that $\partial'_i([w]) = 0$. Therefore, the value of (3) is 0. So, the value of $\partial(g)$ does not depend on the element in $\varphi^{-1}(g)$ we are taking.

Let g be an element of a free (metabelian) Lie algebra. It follows from the definition of derivatives that the value of a partial derivative of g depends on the choice of the system of free generators.

Let S be a metabelian Lie ring and let $g(x_1, x_2, \dots, x_n), f_1(x_1, x_2, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_n)$ be Lie polynomials. Substitute f_1, f_2, \dots, f_n for x_1, x_2, \dots, x_n in $\partial_i g$. By $\partial_i g(f_1, f_2, \dots, f_n)$ denote the obtained expression considered as an element of $U(S)$.

Given the system of elements $\{g_1, g_2, \dots, g_k\}$ of M by $\mathcal{J}(g_1, g_2, \dots, g_k)$ denote the *Jacobi matrix* of this system, i.e. the matrix

$$(\partial_i g_j)_{n \times k} = \begin{pmatrix} \partial_1 g_1 & \partial_1 g_2 & \dots & \partial_1 g_k \\ \partial_2 g_1 & \partial_2 g_2 & \dots & \partial_2 g_k \\ \dots & \dots & \dots & \dots \\ \partial_n g_1 & \partial_n g_2 & \dots & \partial_n g_k \end{pmatrix}.$$

Let

$$f_1(x_1, x_2, \dots, x_n), f_2(x_1, x_2, \dots, x_n), \dots, f_n(x_1, x_2, \dots, x_n)$$

be Lie polynomials. Substitute these polynomials in $\mathcal{J}(g_1, g_2, \dots, g_k)$ for the corresponding x_1, x_2, \dots, x_n and denote by $\mathcal{J}_{f_1, f_2, \dots, f_n}(g_1, g_2, \dots, g_k)$ the obtained matrix.

Let T be the free right $\mathbb{Z}[X]$ -module with a basis t_1, t_2, \dots, t_n . Consider the set \mathcal{M} of square matrices of the second order.

$$\begin{pmatrix} l & 0 \\ \tau & 0 \end{pmatrix},$$

where l is a linear polynomial in $\mathbb{Z}[X]$ and $\tau \in T$. Define the multiplication on \mathcal{M} by the rule $[S_1, S_2] = S_1 S_2 - S_2 S_1$. It is easy to prove that \mathcal{M} is a metabelian Lie ring with respect to this multiplication.

Let ν the homomorphism from M to \mathcal{M} taking each generator x_i to the matrix

$$\begin{pmatrix} y_i & 0 \\ t_i & 0 \end{pmatrix}.$$

It is well-known (see, for example, [1, 2]) that the analogue of ν is an embedding if we consider it as a homomorphism of Lie algebras over a field, for instance, over \mathbb{Q} . Since $\mathbb{Z} \subseteq \mathbb{Q}$, this map is obviously an embedding of M into \mathcal{M} .

By $\mathbb{Z}^1[X]$ denote the set of linear polynomials (without a free term) in variables from the set X . Next, let p, q, m be integers and $m \geq 2$. By $I_{p,q,m}$ denote the ideal in $\mathbb{Z}[X]$ generated by m and $x_i^p(x^q - 1)$, where $i = 1, 2, \dots, n$. Besides, by $\mathbb{Z}_{p,q,m}[X]$ denote the commutative associative ring $\mathbb{Z}[X]/I_{p,q,m}$, by $\mathbb{Z}_m^1[X]$ the set of linear polynomials in $\mathbb{Z}_{p,q,m}[X]$, and by $T_{p,q,m}$ the free right n -generated \mathbb{Z}_m -module. To denote the elements of a basis of $T_{p,q,m}$ we use the characters t_1, t_2, \dots, t_n as well as to denote the elements of the basis of T . Finally, by $\mathcal{M}_{p,q,m}$ we denote the Lie ring of 2×2 matrices of the form

$$\begin{pmatrix} \bar{l} & 0 \\ \bar{\tau} & 0 \end{pmatrix},$$

where $\bar{l} \in \mathbb{Z}_{p,q,m}[X]$, $\bar{\tau} \in T_{p,q,m}$, and the Lie multiplication is defined in the natural way. Obviously, $\mathcal{M}_{p,q,m}$ is a finite metabelian Lie ring.

Let $\eta : \mathbb{Z}[X] \rightarrow \mathbb{Z}_{p,q,m}[X]$ be the natural homomorphism. By $\bar{\partial}_i g$ denote the image of $\partial_i g$ under the action of η .

The following diagram shows the relationship among the rings described above:

$$\begin{array}{ccccccc} L & \xrightarrow{\partial'_i} & U(L) & \xrightarrow{\varphi'} & \mathbb{Z}[X] & \xrightarrow{\eta} & \mathbb{Z}_{p,q,m}[X] \\ \downarrow \varphi & & \nearrow \partial_i & & \nearrow \bar{\partial}_i & & \\ M & & & & & & \end{array}$$

3. PROPERTY OF PRIMITIVITY AND UNIFORM DISTRIBUTION OF ELEMENTS

Let $g \in M$. Later on, by g' we denote the element of M such that $g - g'$ is a linear combination of the elements in X . This linear combination itself will be denoted by \bar{g} . The identities of a free metabelian Lie ring are homogeneous, therefore, for any g the elements \bar{g} and g' are defined uniquely and $g = \bar{g} + g'$. By Δ denote the ideal in $\mathbb{Z}[X]$ generated by the set X . Finally, for any matrix $\mathbf{A} = (a_{ij})_{n \times n}$ we will use the following notation

$$\sigma_i(\mathbf{A}) = \sum_{j=1}^n x_j a_{ji}.$$

We need the following lemmas.

Lemma 3.1. *Let $g \in M$. If $(\partial_1 g, \partial_2 g, \dots, \partial_n g)$ is unimodular, then \bar{g} is primitive in M .*

Proof. It is readily seen that an element of the form $\sum_{i=1}^n m_i x_i$ is primitive in M iff m_1, m_2, \dots, m_n are coprime.

Consider the homomorphism $\varepsilon : \mathbb{Z}[X] \rightarrow \mathbb{Z}$ that takes each polynomial in $\mathbb{Z}[X]$ to its free term. Since $(\partial_1 g, \partial_2 g, \dots, \partial_n g)$ is unimodular, there exists $f_1, f_2, \dots, f_n \in \mathbb{Z}[X]$ such that $\sum_{i=1}^n f_i \partial_i g = 1$. Consequently,

$$\varepsilon\left(\sum_{i=1}^n f_i \partial_i g\right) = \sum_{i=1}^n \varepsilon(f_i) \varepsilon(\partial_i g) = 1.$$

Therefore, $\varepsilon(\partial_1 g), \varepsilon(\partial_2 g), \dots, \varepsilon(\partial_n g)$ are coprime. So, the element $\sum_{i=1}^n \varepsilon(\partial_i g) x_i$ is primitive in M . We are left to note that this element is equal to \bar{g} . \square

Lemma 3.2. *Let \mathbf{A} be an invertible matrix of order n such that its coefficients are in $\mathbb{Z}[X]$. Then $\sigma_1(\mathbf{A}), \sigma_2(\mathbf{A}), \dots, \sigma_n(\mathbf{A})$ generates the ideal Δ .*

Proof. The statement of this lemma follows from the following chain of equalities.

$$(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n) \mathbf{A} \mathbf{A}^{-1} = (\sigma_1(\mathbf{A}), \sigma_2(\mathbf{A}), \dots, \sigma_n(\mathbf{A})) \mathbf{A}^{-1}.$$

□

Lemma 3.3. *Let $\mu : M \rightarrow M$ be an automorphism of M such that $\mu(x_i) = g_i$. Then the Jacobi matrix $\mathcal{J}(g_1, g_2, \dots, g_n)$ is invertible in the ring of matrices over $\mathbb{Z}[X]$.*

Proof. Let μ_1 and μ_2 be endomorphisms of the free metabelian Lie ring M defined as follows: $\mu_j(x_i) = y_{j,i}(x_1, x_2, \dots, x_n)$. Then we have

$$\mu_2 \circ \mu_1(x_i) = \mu_2(\mu_1(x_i)) = y_{1,i}(y_{2,1}(x_1, \dots, x_n), \dots, y_{2,n}(x_1, \dots, x_n)).$$

By $z_j(x_1, x_2, \dots, x_n)$ denote $y_{1,j}(y_{2,1}(x_1, \dots, x_n), \dots, y_{2,n}(x_1, \dots, x_n))$. The following “chain-rule” formulas hold:

$$\partial_i z_j = \sum_{k=1}^n \partial_i y_{2,k} \partial_k y_{1,j}(\bar{y}_{2,1}, \bar{y}_{2,2}, \dots, \bar{y}_{2,n}).$$

These formulas imply

$$(4) \quad \mathcal{J}(z_1, z_2, \dots, z_n) = \mathcal{J}(y_{2,1}, y_{2,2}, \dots, y_{2,n}) \mathcal{J}_{\bar{y}_{2,1}, \bar{y}_{2,2}, \dots, \bar{y}_{2,n}}(y_{1,1}, y_{1,2}, \dots, y_{1,n}).$$

Let μ be an automorphism of M . Then μ^{-1} is also an automorphism. Let $\mu^{-1}(x_i) = f_i$. Using (4) for the identity automorphism we obtain

$$(5) \quad \mathbf{E} = \mathcal{J}(x_1, x_2, \dots, x_n) = \mathcal{J}(g_1, g_2, \dots, g_n) \mathcal{J}_{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n}(f_1, f_2, \dots, f_n).$$

Note that the elements of $\mathcal{J}_{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n}(f_1, f_2, \dots, f_n)$ are polynomials with integer coefficients. It implies that the matrix $\mathcal{J}(g_1, g_2, \dots, g_n)$ is invertible in the ring of matrices over $\mathbb{Z}[X]$. By [16] the Jacobi matrix $\mathcal{J}(g_1, g_2, \dots, g_n)$ is invertible in the ring of matrices over $\mathbb{Q}[X]$. Consequently, the right and left inverses of this matrix coincide. Thus, $\mathcal{J}(g_1, g_2, \dots, g_n)$ is invertible in the ring of matrices over $\mathbb{Z}[X]$. □

Theorem 3.4. *An element $g \in M$ is primitive iff the vector $(\partial_1 g, \partial_2 g, \dots, \partial_n g)$ is unimodular.*

Proof. Suppose that $(\partial_1 g, \partial_2 g, \dots, \partial_n g)$ is unimodular. Lemma 3.1 implies that \bar{g} can be included in some system of generators $\{\bar{g}_1 = \bar{g}, \bar{g}_2, \dots, \bar{g}_n\}$ of the ring $\mathbb{Z}[X]$. By [10] there exists a matrix \mathbf{B} of order n that is invertible in the ring of the matrices over $\mathbb{Z}[X]$ and is such that $\mathbf{B} \cdot (\partial_1 g, \partial_2 g, \dots, \partial_n g)^t = (1, 0, \dots, 0)^t$. So, obviously, the first column of \mathbf{B}^{-1} is equal to $(\partial_1 g, \partial_2 g, \dots, \partial_n g)^t$. Note that $\bar{g} = \sigma_1(\mathbf{B}^{-1})$.

Let R be a subring of $\mathbb{Z}[X]$ such that it is generated by $\bar{g}_2, \bar{g}_3, \dots, \bar{g}_n$. Consider the elements $\sigma_2(\mathbf{B}^{-1}), \sigma_3(\mathbf{B}^{-1}), \dots, \sigma_n(\mathbf{B}^{-1})$. they can be represented as polynomials in $\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n$. Thus, one can find elements $\hat{g}_i \in \mathbb{Z}[X]$ ($i = 2, 3, \dots, n$), such that the element $\sigma_i(\mathbf{B}^{-1}) - \hat{g}_i \sigma_1(\mathbf{B}^{-1})$ can be written as an expression in $\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n$ only. For all $i = 2, 3, \dots, n$ let us subtract the first column of \mathbf{B}^{-1} multiplied by \hat{g}_i from i th column of this matrix. By \mathbf{C} denote the obtained matrix. For $i = 2, 3, \dots, n$ the matrices $\mathbf{E} - \hat{g}_i \mathbf{E}_{1,i}$ are invertible (here \mathbf{E} is the identity $n \times n$ matrix and the matrices $\mathbf{E}_{1,i}$ are the corresponding matrix units). Thus, \mathbf{C} is also invertible because it is a product of \mathbf{B}^{-1} the matrices of the form $\mathbf{E} - \hat{g}_i \mathbf{E}_{1,i}$. The elements $\sigma_1(\mathbf{C}), \sigma_2(\mathbf{C}), \dots, \sigma_n(\mathbf{C})$ generate the same ideal in $\mathbb{Z}[X]$ as the elements $\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n$ do. At it was shown above, this ideal is Δ . Consequently, $\sigma_2(\mathbf{C}), \sigma_3(\mathbf{C}), \dots, \sigma_n(\mathbf{C})$ generate the same ideal R as $\bar{g}_2, \bar{g}_3, \dots, \bar{g}_n$ do.

Consider the following chain of the subrings of R : $\mathbb{Z} \subset \mathbb{Z}[\bar{g}_2] \subset \mathbb{Z}[\bar{g}_2, \bar{g}_3] \subset \dots \subset \mathbb{Z}[\bar{g}_2, \bar{g}_3, \dots, \bar{g}_n] = R$. Easy to see, that this chain satisfies the conditions of Theorem 2.7. Indeed, the first two conditions are obvious (in the second condition

we take \bar{g}_{i+1} for y_i), the third condition follows from [10]. The fourth condition is also obvious because $I_r = \langle \bar{g}_2, \bar{g}_3, \dots, \bar{g}_{r+1} \rangle$, therefore I_r/I_r^2 is isomorphic to the set of linear combinations of $\bar{g}_2, \bar{g}_3, \dots, \bar{g}_{r+1}$. It means that I_r/I_r^2 is a \mathbb{Z} -module of rank r . Consequently, the set of all $(n-1) \times (n-1)$ -matrices with coefficients in $\mathbb{Z}[\bar{g}_2, \bar{g}_3, \dots, \bar{g}_n]$ acts transitively on the set of all I_{n-1} -modular vectors. In other words, there exists an invertible matrix \mathbf{D}_1 (in which the elements $\bar{g}_2, \bar{g}_3, \dots, \bar{g}_n$ are written as expressions in x_1, x_2, \dots, x_n), such that $(\sigma_2(\mathbf{C}), \sigma_3(\mathbf{C}), \dots, \sigma_n(\mathbf{C}))\mathbf{D}_1 = (\bar{g}_2, \bar{g}_3, \dots, \bar{g}_n)$.

Let

$$\mathbf{D} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \mathbf{D}_1 \end{array} \right).$$

Then

$$(x_1, x_2, \dots, x_n)\mathbf{CD} = (\sigma_1(\mathbf{C}), \sigma_2(\mathbf{C}), \dots, \sigma_n(\mathbf{C}))\mathbf{D} = (\bar{g}_1, \bar{g}_2, \dots, \bar{g}_n).$$

Therefore, we obtain $\sigma_i(\mathbf{CD}) = \bar{g}_i$ for all $i = 1, 2, \dots, n$. There is a unique representation

$$(6) \quad \mathbf{CD} = \bar{\mathbf{F}} + \mathbf{F}',$$

where $\bar{\mathbf{F}}$ is such that its elements are integers and \mathbf{F}' is such that its elements are polynomials in the variables from the set X without free terms.

By computing degrees of polynomials in both sides of (6) one can obtain that $\sigma_i(\bar{\mathbf{F}}) = \bar{g}_i$. Therefore, $\sigma_i(\mathbf{F}') = 0$. Thus, [16] implies that there exist elements $g'_1, g'_2, \dots, g'_n \in M'_{\mathbb{Q}}$ such that $\mathbf{CD} = \mathcal{J}(g_1, g_2, \dots, g_n)$, where $g_i = \bar{g}_i + g'_i$ for $i = 1, 2, \dots, n$.

Let $[u] = [\dots [x_{i_1}, x_{i_2}]x_{i_3}] \dots x_{i_k} \in M'$. It is easy to show by induction that

$$(7) \quad \partial_i[u] = \begin{cases} x_{i_2}x_{i_3} \dots x_{i_k}, & \text{if } i = i_1 \\ -x_{i_1}x_{i_3} \dots x_{i_k}, & \text{if } i = i_2 \\ 0, & \text{else.} \end{cases}$$

Let us put a linear order on X . By [4, 9], the set of all right-normed elements of the form $[\dots [x_{i_1}, x_{i_2}]x_{i_3}] \dots x_{i_k}$, where $x_{i_2} < x_{i_1}$, $x_{i_2} \leq x_{i_3} \leq \dots \leq x_{i_k}$ is a basis of M . Later on, such monomials will be called *basis* ones.

Suppose that $h = \alpha[u]$ for some $[u] \in M'$ starting with x_i . It follows from (7) that $\partial_i h = \alpha \partial_i [u]$. Easy to see that for distinct monomials $[u]$ the non-zero values of $\partial_i [u]$ are also distinct. Let h be an element of M' such that $h = \sum \alpha_j [u_j]$, where $[u_j] \in M'$ and let $[u_j]$ start with x_{i_j} . If we write derivatives $\partial_i h$ as a linear combinations of $\partial_i [u_j]$ then we can see that for any j the monomial $\partial_{i_j} [u_j]$ occurs only once in the obtained sum. So, the coefficient by $\partial_{i_j} [u_j]$ in $\partial_i h$ is equal to α_j .

We have shown that all derivatives of g'_1, g'_2, \dots, g'_n are the polynomials with integer coefficients. Represent g'_1, g'_2, \dots, g'_n as linear combinations of basis monomials. By the last paragraph, all coefficients by these monomials are.

Since \mathbf{C} and \mathbf{D} are invertible so is their product. Again by [16] we obtain that the homomorphism $\mu : M \rightarrow M$ defined by the rule $\mu(x_i) = g_i$ is the automorphism. So, $\{g_1, g_2, \dots, g_n\}$ is the set of free generators of M .

Conversely, let g be primitive. Then it can be included in a system of free generators $g = g_1, g_2, \dots, g_n$ of M . So, $\mu : M \rightarrow M$ defined by the rule $\mu(x_i) = g_i$ is an automorphism. Therefore, Lemma 3.3 implies $|\mathcal{J}(g_1, g_2, \dots, g_n)| = \pm 1$. Expanding this determinant by the first column we obtain an expression of the form $\sum_{i=1}^n s_i \partial_i g = \pm 1$, where $s_i \in \mathbb{Z}[X]$ for $i = 1, 2, \dots, n$. Multiplying by -1 if necessary, we see that 1 lies in the ideal in the ring $\mathbb{Z}[X]$ that is generated by the derivatives of g . This completes the proof. \square

Lemma 3.5. *An element $g \in M$ is primitive iff for any integer numbers p, q and for any integer positive number $m \geq 2$ the vector $(\bar{\partial}_1 g, \bar{\partial}_2 g, \dots, \bar{\partial}_n g)$ is unimodular in $\mathbb{Z}_{p,q,m}[X]$.*

Proof. If g is primitive then the vector $(\partial_1 g, \partial_2 g, \dots, \partial_n g)$ is unimodular by Lemma 3.4. It means that the ideal generated by the coordinates of this vector coincides with the entire ring. It is clear that for any ring $\mathbb{Z}_{p,q,m}[X]$ the natural homomorphism $\mathbb{Z}[X] \rightarrow \mathbb{Z}_{p,q,m}[X]$ is surjective. Therefore, the vector $(\bar{\partial}_1 g, \bar{\partial}_2 g, \dots, \bar{\partial}_n g)$ is also unimodular.

Conversely, let $g \in M$ be not primitive. Then, by Theorem 3.4 the vector $(\partial_1 g, \partial_2 g, \dots, \partial_n g)$ is not unimodular. It means that the coordinates of this vector generate an ideal that does not coincide with the entire ring $\mathbb{Z}[X]$ or equivalently, $\mathbb{Z}[X]/I$ is non-trivial. Obviously, this ring is finitely generated. So, [3] implies that there exists an ideal J such that $R = \mathbb{Z}[X]/I + J$ is a non-trivial finite ring. Consider the images $\check{x}_1, \check{x}_2, \dots, \check{x}_n$ of the elements x_1, x_2, \dots, x_n under the natural homomorphism $\mathbb{Z}[X] \rightarrow R$. Since the ring R is finite, for any \check{x}_i there are integers $p_i, q_i > 0$ such that $\check{x}_i^{p_i} = \check{x}_i^{q_i + q_i}$. Take p_i to be least possible and for each such p_i take q_i to be least possible. Moreover, there exists a positive integer number $m \geq 2$ such that $m = 0$ in R . Take m to be also least possible. Let $p = \max\{p_1, p_2, \dots, p_n\}$ and q be the least common multiple of q_1, q_2, \dots, q_n .

It is clear that the map $X \rightarrow R$ defined by the rule $x_i \rightarrow \check{x}_i$ can be extended up to the homomorphism $\theta : \mathbb{Z}_{p,q,m}[X] \rightarrow R$. Indeed, since $\mathbb{Z}[X]$ is the free associative commutative n -generated ring and $\mathbb{Z}_{p,q,m}[X] = \mathbb{Z}[X]/I_{p,q,m}$ we are only left to show that θ is well-defined. It means that the relations generating $I_{p,q,m}$ go to 0 zero by the action of θ . Let us verify it for all relations generating $I_{p,q,m}$. We have $\theta(m) = m = 0$ by the choice of m . Since q is the least common multiple of q_1, q_2, \dots, q_n it divides all q_i . Therefore, $q = q_i \tilde{q}_i$ for some positive integer number \tilde{q}_i ($i = 1, 2, \dots, n$). Consequently,

$$\begin{aligned} \theta(x_i^p(x_i^q - 1)) &= \check{x}_i^p(\check{x}_i^q - 1) = \\ &= \check{x}_i^{p-p_i} \check{x}_i^{p_i} (\check{x}_i^{q_i \tilde{q}_i} - 1) = \\ &= \check{x}_i^{p-p_i} (\check{x}_i^{q_i(\tilde{q}_i-1)} + \check{x}_i^{q_i(\tilde{q}_i-2)} + \dots + \check{x}_i^{q_i} + 1)(\check{x}_i^{p_i}(\check{x}_i^{q_i} - 1)) = \\ &= 0 \end{aligned}$$

Thus, θ is a homomorphism and it is clearly surjective. The natural homomorphism from $\mathbb{Z}[X]$ to $\mathbb{Z}_{p,q,m}[X]$ takes the derivatives of g to $\bar{\partial}_1 g, \bar{\partial}_2 g, \dots, \bar{\partial}_n g$ correspondingly. Moreover, the ideal generated by these elements does not coincide with the entire ring $\mathbb{Z}_{p,q,m}[X]$. Indeed, on the one hand, θ is a surjective homomorphism, but on the other hand, all elements $\bar{\partial}_i g$ lie in the kernel of θ . Therefore, if $(\bar{\partial}_1 g, \bar{\partial}_2 g, \dots, \bar{\partial}_n g)$ were unimodular then R would be equal to 0. We get a contradiction. \square

Lemma 3.6. *If $g \in M$ be uniformly distributed on the variety of metabelian Lie rings, then the natural homomorphism from M to M/M' takes g to a primitive element on the variety of free abelian Lie rings.*

Proof. Suppose that $g \in M'$. Let A be an arbitrary finite abelian Lie ring. Let us choose a basis of M consisting of Lie monomials and represent g as an integer valued linear combination of the elements of this basis. Obviously, any monomial in this linear combination is a product of at least two generators of M . Therefore, $g(r_1, r_2, \dots, r_n) = 0$ in A for any $r_1, r_2, \dots, r_n \in A$. But $g(x_1, x_2, \dots, x_n)$ is uniformly distributed on the variety of metabelian Lie rings. In particular, it is uniformly distributed on any finite abelian Lie ring. This is a contradiction. So, $g \in M \setminus M'$.

Arguing as in the last paragraph we see that $g'(r_1, r_2, \dots, r_n) = 0$ for any $r_1, r_2, \dots, r_n \in A$. Consequently, for any $r_1, r_2, \dots, r_n \in A$ we have $g(r_1, r_2, \dots, r_n) = \bar{g}(r_1, r_2, \dots, r_n)$. Therefore, the element \bar{g} is uniformly distributed on A .

Note that the image of g by the natural homomorphism $M \rightarrow M/M'$ is equal to \bar{g} . Since we can consider an arbitrary the finite abelian Lie ring A we obtain \bar{g} is uniformly distributed on the variety of abelian Lie ring. Since the multiplication on abelian Lie rings is trivial, we can consider an abelian Lie ring as an additive abelian group. Since \bar{g} is uniformly distributed on the variety of abelian groups, Lemma 2.4 implies that \bar{g} is primitive in the free abelian group G generated by the set X . It means that this element is primitive in the free abelian Lie ring that is obtained from G by adding on the multiplication in the trivial way. This is exactly what we needed. \square

We need one more well known auxiliary lemma. We are going to give its proof for the completeness of reasoning.

Lemma 3.7. *Let*

$$S_1 = \begin{pmatrix} s_1 & 0 \\ \tau_1 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} s_2 & 0 \\ \tau_2 & 0 \end{pmatrix}, \dots, S_n = \begin{pmatrix} s_n & 0 \\ \tau_n & 0 \end{pmatrix}$$

be elements of the Lie ring $\mathcal{M}_{p,q,m}$. Then

$$(8) \quad g(S_1, S_2, \dots, S_n) = \begin{pmatrix} \bar{g}(s_1, s_2, \dots, s_n) & 0 \\ \tau' & 0 \end{pmatrix},$$

where $\tau' = \sum_{i=1}^n \tau_i \cdot \bar{\partial}_i g(s_1, s_2, \dots, s_n)$.

Proof. Let $g = [v]$. Without loss of generality we may assume that $[v]$ is a right-normed monomial, i.e. a monomial of the form $[\dots [x_{i_1}, x_{i_2}], \dots, x_{i_k}]$. If $\ell([v]) = 1$ then the proof is trivial. Let $\ell([v]) \geq 2$. Suppose that the statement is true for any monomials of lesser length. We have $[v] = [[u], x_j]$. By induction hypothesis

$$[u](S_1, S_2, \dots, S_n) = \begin{pmatrix} \overline{[u]}(s_1, s_2, \dots, s_n) & 0 \\ \tau'_{[u]} & 0 \end{pmatrix},$$

where

$$(9) \quad \tau'_{[u]} = \sum_{i=1}^n \tau_i \cdot \bar{\partial}_i [u](s_1, s_2, \dots, s_n).$$

Let us notice that

$$\overline{[u]}(s_1, s_2, \dots, s_n) = \begin{cases} s_i, & \text{if } [u] = x_i, \ i = 1, 2, \dots, n \\ 0, & \text{if } \ell([u]) \geq 2 \end{cases}.$$

For any $\bar{\partial}_i$ the following equality holds:

$$(10) \quad \begin{aligned} \bar{\partial}_i [v] &= \bar{\partial}_i [[u], x_j] = \\ &= \eta \circ \varphi'(\partial'_i [u] \cdot x_j - \partial'_i x_j \cdot [u]) = \\ &= \eta(\partial_i [u] \cdot x_j - \partial_i x_j \overline{[u]}) = \\ &= \bar{\partial}_i [u] \cdot x_j - \delta_{ij} \overline{[u]} \end{aligned}$$

where δ_{ij} is Kronecker delta. On the other hand,

$$(11) \quad \left[\begin{pmatrix} \overline{[u]}(s_1, s_2, \dots, s_n) & 0 \\ \tau'_{[u]} & 0 \end{pmatrix}, \begin{pmatrix} s_j & 0 \\ \tau'_j & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ \tau'_{[u]} s_j - \tau_j \overline{[u]}(s_1, s_2, \dots, s_n) & 0 \end{pmatrix}.$$

By (3), (9) and (11) we get

$$\tau' = \left(\sum_{i=1}^n \tau_i \cdot \bar{\partial}_i[u](s_1, s_2, \dots, s_n) \right) s_j - \tau_j \overline{[u]}(s_1, s_2, \dots, s_n).$$

Taking into account (10), we obtain $\tau' = \sum_{i=1}^n \tau_i \cdot \bar{\partial}_i[u](s_1, s_2, \dots, s_n)$. So, for the case of a Lie monomial the proof is complete. If g is a Lie polynomial, then the statement follows from the linearity of $\bar{\partial}_i$ and the linearity of the matrix addition. \square

Theorem 3.8. *An element $g \in M$ is primitive iff it is uniformly distributed on the variety of metabelian Lie algebras.*

Proof. Let g be primitive. Then it can be included in a set of generators of M . As we have shown, the property of an element to be uniformly distributed on the variety of metabelian Lie algebras does not depend on a set of free generators chosen in M . Therefore, we may assume that $g \in X$, for instance, $g = x_1$. Let R be an arbitrary finite metabelian Lie ring. Then for any elements r_1, r_2, \dots, r_n we have $g(r_1, r_2, \dots, r_n) = r_1$. It means that g takes any fixed value for $|R|^{n-1}$ n -tuples $(r_1, r_2, \dots, r_n) \in R^n$. Therefore, g is uniformly distributed on R . Since finite metabelian Lie ring R has been chosen arbitrarily we obtain that g is uniformly distributed on the variety of metabelian Lie rings.

Let g be uniformly distributed on the variety of metabelian Lie rings. As we have noticed above, \bar{g} is the image of g under the natural map $M \rightarrow M/M'$. By Lemma 3.6 we obtain that \bar{g} is primitive in M/M' . Easy to show, that in abelian Lie ring a linear basis is a set of generators. Therefore, without loss of generality it can be assumed that $\bar{g} \in X$ (say, $\bar{g} = x_1$).

Consider a ring $\mathcal{M}_{p,q,m}$. Lemma 3.7 implies that for arbitrary elements

$$S_1 = \begin{pmatrix} s_1 & 0 \\ \tau_1 & 0 \end{pmatrix}, S_2 = \begin{pmatrix} s_2 & 0 \\ \tau_2 & 0 \end{pmatrix}, \dots, S_n = \begin{pmatrix} s_n & 0 \\ \tau_n & 0 \end{pmatrix}$$

of this ring the following equality holds:

$$g(S_1, S_2, \dots, S_n) = \begin{pmatrix} \bar{g}(s_1, s_2, \dots, s_n) & 0 \\ \tau' & 0 \end{pmatrix},$$

where $\tau' = \sum_{i=1}^n \tau_i \cdot \bar{\partial}_i g(s_1, s_2, \dots, s_n)$. Since g is uniformly distributed on the variety of metabelian Lie algebras it is uniformly distributed on any ring, in particular,

on $\mathcal{M}_{p,q,m}$. Consequently, for any matrix $Q = \begin{pmatrix} a & 0 \\ \tilde{\tau} & 0 \end{pmatrix}$ the equation

$$g(S_1, S_2, \dots, S_n) = Q$$

has exactly $|\mathcal{M}_{p,q,m}|^{n-1}$ solutions. Therefore the system of equations

$$(12) \quad \begin{cases} \bar{g}(s_1, s_2, \dots, s_n) = a \\ \sum_{i=1}^n \tau_i \cdot \bar{\partial}_i g(s_1, s_2, \dots, s_n) = \tilde{\tau} \end{cases}$$

has $|\mathcal{M}_{p,q,m}|^{n-1}$ solutions. Here we say that the solution of system (12) is a vector

$$(s_1, s_2, \dots, s_n, \tau_1, \tau_2, \dots, \tau_n) \in \mathbb{Z}_m^1[X]^n \times T^n$$

satisfying both equations of this system.

Let $T^n = T \oplus T \oplus \dots \oplus T$. For any elements $s_1, s_2, \dots, s_n \in \mathbb{Z}_m^1[X]$ consider the map $\xi_{s_1, s_2, \dots, s_n} : T^n \rightarrow T$ defined by the rule

$$\xi_{s_1, s_2, \dots, s_n}(\tau_1, \tau_2, \dots, \tau_n) = \sum_{i=1}^n \tau_i \cdot \bar{\partial}_i g(s_1, s_2, \dots, s_n).$$

Clearly, this map is a homomorphism of $\mathbb{Z}_{m,p,q}$ -modules. Therefore, cardinality of $\text{Ker } \xi_{s_1, s_2, \dots, s_n}$ is not less than $|T|^{n-1}$. Moreover, if it is equal to $|T|^{n-1}$, then $\xi_{s_1, s_2, \dots, s_n}$ is a surjective homomorphism. Thus, for any s_1, s_2, \dots, s_n and for any $\tilde{\tau}$ in the image of $\xi_{s_1, s_2, \dots, s_n}$ the second equation of system (12) has at least $|T|^{n-1}$ solutions $(\tau_1, \tau_2, \dots, \tau_n)$.

Consider system (12) for $a = x_1, \tilde{\tau} = 0$. Since $g = x_1 + g'$, any solution of the first equation of system (12) has the form $(x_1, s_2, s_3, \dots, s_n)$, and any n -tuple of this form is a solution of system (12). So, cardinality of the set of n -tuples satisfying the first equation is $|\mathbb{Z}_m^1[X]|^{n-1}$. As we have shown above, for each such n -tuple, there are at least $|T|^{n-1}$ n -tuples $(\tau_1, \tau_2, \dots, \tau_n)$ satisfying the second equation. We are left to notice that $|\mathcal{M}_{m,p,q}| = |\mathbb{Z}_m^1[X]| \cdot |T|$. Therefore, if for some n -tuple $(x_1, s_2, s_3, \dots, s_n)$ there are more than $|T|^{n-1}$ solutions then the cardinality of the set of all solutions of system (12) is greater than $|\mathbb{Z}_m^1[X]|^{n-1} \cdot |T|^{n-1} = |\mathcal{M}_{m,p,q}|^{n-1}$. This contradicts to the assumption that g is uniformly distributed on $\mathcal{M}_{m,p,q}$.

So, if $\tilde{\tau} = 0$ then for any n -tuple of the form $(x_1, s_2, s_3, \dots, s_n)$ there are exactly $|T|^{n-1}$ n -tuples $(\tau_1, \tau_2, \dots, \tau_n)$ satisfying the second equation of system (12). Consequently, for any n -tuple of the form $(x_1, s_2, \dots, s_n) \in \mathbb{Z}_m^1[X]$ the map $\xi_{x_1, s_2, \dots, s_n}$ is a surjective homomorphism. So, for any such n -tuple the equation $\sum_{i=1}^n \tau_i \cdot \bar{\partial}_i g(x_1, s_2, \dots, s_n) = \tilde{\tau}$ has a solution. In particular, the equation

$$(13) \quad \sum_{i=1}^n \tau_i \cdot \bar{\partial}_i g(x_1, x_2, \dots, x_n) = t_1.$$

has a solution.

Let $\tau_i = \sum_{j=1}^n t_j \alpha_{ij}$ for $i = 1, 2, \dots, n$. Then, considering the coefficients by t_1 in the right-hand side and in the left-hand side we obtain

$$\sum_{i=1}^n \alpha_{i,1} \bar{\partial}_i g(x_1, x_2, \dots, x_n) = 1.$$

Therefore, the vector $(\bar{\partial}_1 g, \bar{\partial}_2 g, \dots, \bar{\partial}_n g)$ is unimodular in $\mathbb{Z}_{m,p,q}$. Since this holds for any numbers m, p, q , Lemma 3.5 implies that g is primitive. \square

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